1.1. Differential forms.

a) Show that a closed n-form ω on the unit sphere $S^n \subset \mathbb{R}^{n+1}$ is exact if and only if $\int_{S^n} \omega = 0$.

b) Prove that a compactly supported n-form ω on $\mathbb{R}^n$ is the exterior derivative of a compactly supported $(n-1)$-form on $\mathbb{R}^n$ if and only if $\int_{\mathbb{R}^n} \omega = 0$.
Hint: Use stereographic projection to carry ω to a form ω' on S^n. By part a), $\omega' = d\nu$. Now $d\nu$ is 0 in a contractible neighborhood U of the north pole N. Use this to find an $n-2$ form μ on S^n such that $\nu = d\mu$ near N. Then $\nu - d\mu$ is zero near N, so it pulls back to a compactly supported form on $\mathbb{R}^n$.

Solution. a) By Stokes' Theorem, integration yields a well-defined linear map

$$I\colon H^n_{\mathrm{dR}}(S^n) \to \mathbb{R}.$$

Since the integral of a volume form is non-zero, I is surjective. Furthermore, as seen in class, $H^n_{\mathrm{dR}}(S^n)$ is 1-dimensional, so I is an isomorphism. The kernel, the class $[0]$, is represented precisely by the exact forms.
b) The hint gives essentially the full solution. □

1.2. De Rham cohomology of T^2. Determine the De Rham cohomology of the torus T^2.

Solution. Cover the torus T^2 using two (sets homeomorphic to) cylinders U, V so that they overlap slightly at the extremities. Then U and V are both homotopy equivalent to S^1, so

$$H^s_{dR}(U) \cong H^s_{dR}(V) \cong H^s_{dR}(S^1) \cong \begin{cases} \mathbb{R} & s = 0, 1, \\ 0 & s \geq 2. \end{cases}$$

The intersection $U \cap V$ is homotopy equivalent to the disjoint union of two copies of S^1 and it is not difficult to see (e.g by using the Mayer-Vietoris sequence) that

$$H^s_{dR}(U \cap V) \cong H^s_{dR}(S^1 \sqcup S^1) \cong H^s_{dR}(S^1) \oplus H^s_{dR}(S^1) \cong \begin{cases} \mathbb{R}^2 & s = 0, 1, \\ 0 & s \geq 2. \end{cases}$$

Using that $H^0_{dR}(T^2) \cong \mathbb{R}$ and, as seen in the lecture, $H^2_{dR}(T^2) \cong \mathbb{R}$, the Mayer-Vietoris sequence for this open cover is

$$0 \to H^0_{dR}(T^2) \to H^0_{dR}(U) \oplus H^0_{dR}(V) \to H^0_{dR}(U \cap V) \to H^1_{dR}(T^2)$$

$$\to H^1_{dR}(U) \oplus H^1_{dR}(V) \to H^1_{dR}(U \cap V) \to H^2_{dR}(T^2) \to 0,$$

which amounts to

$$0 \to \mathbb{R} \to \mathbb{R}^2 \to \mathbb{R}^2 \to H^1_{dR}(T^2) \to \mathbb{R}^2 \to \mathbb{R}^2 \to \mathbb{R} \to 0.$$

Since the alternating sum of the dimensions is 0, we conclude that $-1 + 2 - 2 + \dim H^1_{dR}(T^2) - 1 + 2 - 1 = 0$, and therefore $H^1_{dR}(T^2) \cong \mathbb{R}^2$. □

1.3. Tensor fields. Let T be a $(1,2)$-tensor field on M^m. Let (φ, U) and (ψ, U) be two charts on M. Show that the component ${}^{\psi}T^c_{ab}$ of T with respect to ψ depends on the components ${}^{\varphi}T^k_{ij}$ of T with respect to φ by the following relation:

$${}^{\psi}T^c_{ab} = \sum_{i,j,k=1}^{m} \frac{\partial \psi^c}{\partial \varphi^k} \frac{\partial \varphi^i}{\partial \psi^a} \frac{\partial \varphi^j}{\partial \psi^b} \; {}^{\varphi}T^k_{ij}.$$

Solution. For $a, b, c \in \{1, \dots, m\}$, we have

$${}^{\psi}T^c_{ab} = T(d\psi^c \otimes \frac{\partial}{\partial \psi^a} \otimes \frac{\partial}{\partial \psi^b})$$

so by writing T with respect to φ we see that

$$\begin{aligned} {}^{\psi}T^c_{ab} &= \sum_{i,j,k=1}^{m} {}^{\varphi}T^k_{ij} \frac{\partial}{\partial \varphi^k} \otimes d\varphi^i \otimes d\varphi^j \Big(d\psi^c \otimes \frac{\partial}{\partial \psi^a} \otimes \frac{\partial}{\partial \psi^b} \Big) \\ &\sum_{i,j,k=1}^{m} {}^{\varphi}T^k_{ij} \Big(d\psi^c (\frac{\partial}{\partial \varphi^k}) \Big) \Big(d\varphi^i (\frac{\partial}{\partial \psi^a}) \Big) \Big(d\varphi^j (\frac{\partial}{\partial \psi^b}) \Big). \end{aligned}$$

Since

$$d\psi^c (\frac{\partial}{\partial \varphi^k}) = \frac{\partial \psi^c}{\partial \varphi^k}, \quad d\varphi^i (\frac{\partial}{\partial \psi^a}) = \frac{\partial \varphi^i}{\partial \psi^a}, \quad d\varphi^j (\frac{\partial}{\partial \psi^b}) = \frac{\partial \varphi^j}{\partial \psi^b},$$

substituting these into the sum gives

$${}^{\psi}T^c_{ab} = \sum_{i,j,k=1}^{m} {}^{\varphi}T^k_{ij} \frac{\partial \psi^c}{\partial \varphi^k} \frac{\partial \varphi^i}{\partial \psi^a} \frac{\partial \varphi^j}{\partial \psi^b}.$$

□